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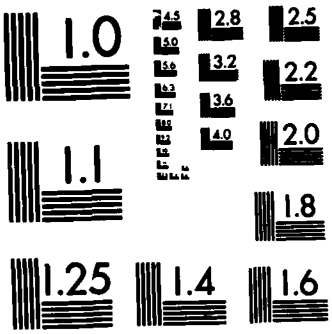


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THE DISTRIBUTIONS OF MEASURES OF VISIBILITY

ON LINE SEGMENTS IN THREE DIMENSIONAL

SPACES UNDER POISSON SHADOWING PROCESSES

by

M. Yadin and S. Zacks

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# THE DISTRIBUTIONS OF MEASURES OF VISIBILITY ON LINE SEGMENTS IN THREE DIMENSIONAL SPACES UNDER POISSON SHADOWING PROCESSES

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Abstract

The problem of determining the moments of visibiltiy measures on star-shaped curves, under Poisson random shadowing process in the plane (see [ 4 ]) is generalized here to the problem of determining the moments of visibility measures on line segments in a three-dimensional space, when the shadowing process is created by a Poisson random field of spheres of random size. The present paper provides the methodology for reducing the three-dimensional problem to a two-dimensional one, and employing the general theory of [4 ] to solve the present problem. Although the general approach is similar, the functions derived in the present paper are somewhat different for the sake of simplifying the computations.

Key Words: *Poisson Shadowing process; Lines of Sight; Visibility Probabilities; Measures of Visibility; Moments of Visibility.*

## 1. Introduction

Consider a source of light located in the origin and an arbitrary straight line segment, $C$, in a three dimensional Euclidean space, $R^3$. A layer of random spheres is located between the origin and $C$. The centers of the spheres are randomly located between two parallel planes, according to a Poisson process with a specified intensity. Furthermore, the radii of the spheres are independent identically distributed random variables, having a specified distribution, such that both the origin, $O$, and the line segment, $C$, are not covered by the random spheres. A point $P$ on $C$ is called visible (in the light), if the line segment $\overline{OP}$ does not intersect any random sphere. The measure of visibility on $C$ is the total portion of $C$ which is visible. We are interested in the distribution of this random variable.

In a previous study [ 4 ] we developed the methodology of determining the moments of the measure of visibility on star-shaped curves in the plane and provided also an approximation to its distribution. The objective of the present paper is to develop a similar methodology for treating three dimensional problems. For this purpose we devote Section 2 to the discussion of the geometry of the three dimensional shadowing model. The three dimensional problem is reduced in Section 3 to a two dimensional problem. We distinguish between two cases: the case where the plane containing $O$ and $C$ is perpendicular to the layer of the random spheres and the case where that plane is slanted with respect to that layer. In the first case, a standard Poisson field of spheres (uniformly distributed centers with radii independent of the location of the centers) is reduced to a standard Poisson field of (intersecting) disks on the plane. This is not the case, however, when the plane is slanted. Subsections 3.1 and 3.2 are devoted to these two cases, the standard and the non-standard ones. In Section 4 we introduce the notion of visibility measure and provide recursive formulae for computing its moments. Section 5 provides a development of certain basic functions, called here the K-functions, required for the determination of visibility probabilities and for the moments of visibility measure. In Section 6 we further derive the K-functions under the assumption that the radii of the spheres are uniformly distributed over the interval $[0,b]$. We probide also some numerical illustrations of the standardized moments of the measure of visibility. Derivation of the first two moments of

the distribution of the radii of disks in the plane intersecting the layer of spheres, and derivation of the corresponding non-central moments is provided in appendices. The theory provided here has a variety of important applications. For example, a helicopter is flying over a forest, and its flight path, at a specified time period, can be described by a given vector. An observer is located at a given point in the forest. The crowns of trees are located at different heights, and can be modeled (approximately) as spheres of random radii. The problem is to determine the distribution of the visible portion of the flight path. Instead of helicopter and trees in the forest, one can think of an airplane and a layer of clouds. There are many other applications in areas of communication with sattelites, astronomy, biology, fishery, etc.

## 2. The Three Dimensional Shadowing Model

Let $U^*$, $W^*$ and $C^*$ be three parallel planes, having distances $u^*$, $w^*$ and $r^*$, respectively, from a point $0$. It is assumed that

$$0 < b \le u^* < w^* \le r^* - b \quad , \tag{2.1}$$

where $b$ is some specified positive value. Let $C$ be a straight line in $C^*$ and $M$ the plane passing through $0$ and $C$. Let $W$ and $U$ be the straight line in which $M$ intersects $W^*$ and $U^*$, respectively. The three parallel lines $U$, $W$ and $C$ are at distances u, w and r from $0$, satisfying the relations

$$\frac{u^*}{u} = \frac{w^*}{w} = \frac{r^*}{r} = \cos\phi \quad , \tag{2.2}$$

where $\phi$ is the angle between the axes $Z^*$ and $Z$ (see Fig. 1). Notice that $Z^*$ is perpendicular to $C^*$, while $Z$ lies on $M$ and is perpendicular to $C$ and both $Z$ and $Z^*$ pass through $0$.

A source of light is located at $0$ and spheres of random diameters are randomly centered between $U^*$ and $W^*$. It is assumed that the centers of the spheres follow a homogeneous Poisson process of intensity $\lambda$, between $U^*$ and $W^*$. Furthermore, the radii of the spheres are independent, identically distributed (iid) random variables, $X_1, X_2, \ldots$ having a common c.d.f. $F(x)$ concentrated on $[0,b]$. Condition (2.1) implies that none of these random spheres either covers $0$ or intersects $C^*$. These spheres cast shadows on $C^*$ and in particular on $C$.

For the purpose of studying the visibility on the line $C$ we consider the two-dimensional shadowing process on $M$. Shadows on $C$ are cast by random disks on $M$, which are generated by random spheres (intersecting $M$).

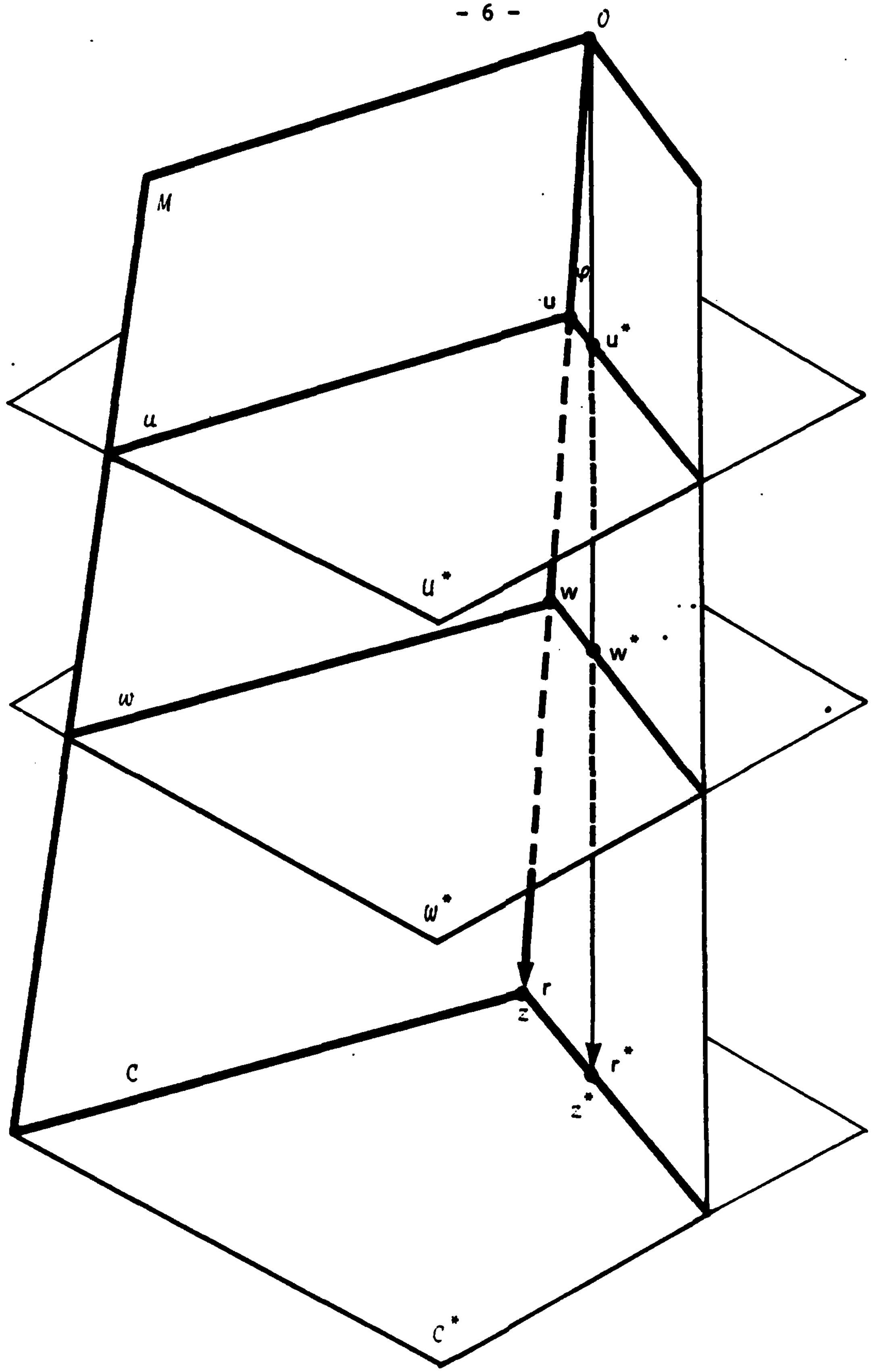


Fig. 1. The Geometry of the Three Dimensional Model.

## 3. Reduction to a Two-Dimensional Shadowing Model

In the present section we consider the reduction of the shadowing model to $M$. In particular, we have to determine the distribution of the centers of shadowing disks on $M$ and the conditional distribution of their radii given the location of their centers. In Fig. 2 we present the geometry of the reduction to a two-dimensional model by the projection of the three-dimensional space on a plane perpendicular to $C$. We restrict attention to spheres which are centered within a prism, $P$, whose projection is the parallelogram ABCD (see Fig. 2). The distances from $M$ of spheres centered outside this prism is greater than b. Accordingly, such spheres do not intersect $M$ and therefore they cannot cast shadows on $C$. A sphere centered within the prism $P$ intersects $M$ if its radius, x, is greater than its distance from $M$, d. The intersection of such a sphere with $M$ is a disk of radius $y = (x^2-d^2)^{1/2}$. For simplicity, we will assume that all the spheres centered within the prism $P$ generate on $M$ disks; those spheres with $x \le d$ generate on $M$ a disk with radius $y=0$. The center of a disk generated by a sphere is the projection of its center on $M$. Such projections may lie between two parallel lines on $M$, having distances $u-\beta$ and $w+\beta$ from 0, where

$$\beta = b \tan \phi \quad . \tag{3.1}$$

### 3.1 Random Disks on $M$ When $\beta = 0$

In the case of $\beta=0$ ($\phi=0$) the plane $M$ is perpendicular to the planes $C^*$, $W^*$ and $U^*$. In this case $\frac{r}{r^*} = \frac{w}{w^*} = \frac{u}{u^*} = 1$. The distribution of the location of the center of disks on $M$ follows in the present case a homogeneous Poisson process, with intensity $\mu=2\lambda b$, between two parallel lines $U$ and $W$ on $M$, having distances u and w from 0. Indeed, consider any Borel set, B, on $M$, of area A located between $U$ and $W$. Disks which are centered in B are generated by spheres centered in the cylinder set of B within the prism $P$. The volume of this cylinder set is 2bA. Accordingly, the number $N\{B\}$ of disks on $M$ centered in B has a Poisson distribution with mean $2\lambda bA$.

Furthermore, since centers of spheres in $P$ are uniformly distributed, the distances of these centers from $M$ are i.i.d. random variables, having a common uniform distribution on $[0,b]$. Accordingly, the c.d.f. of the radius of a disk on $M$, Y, is

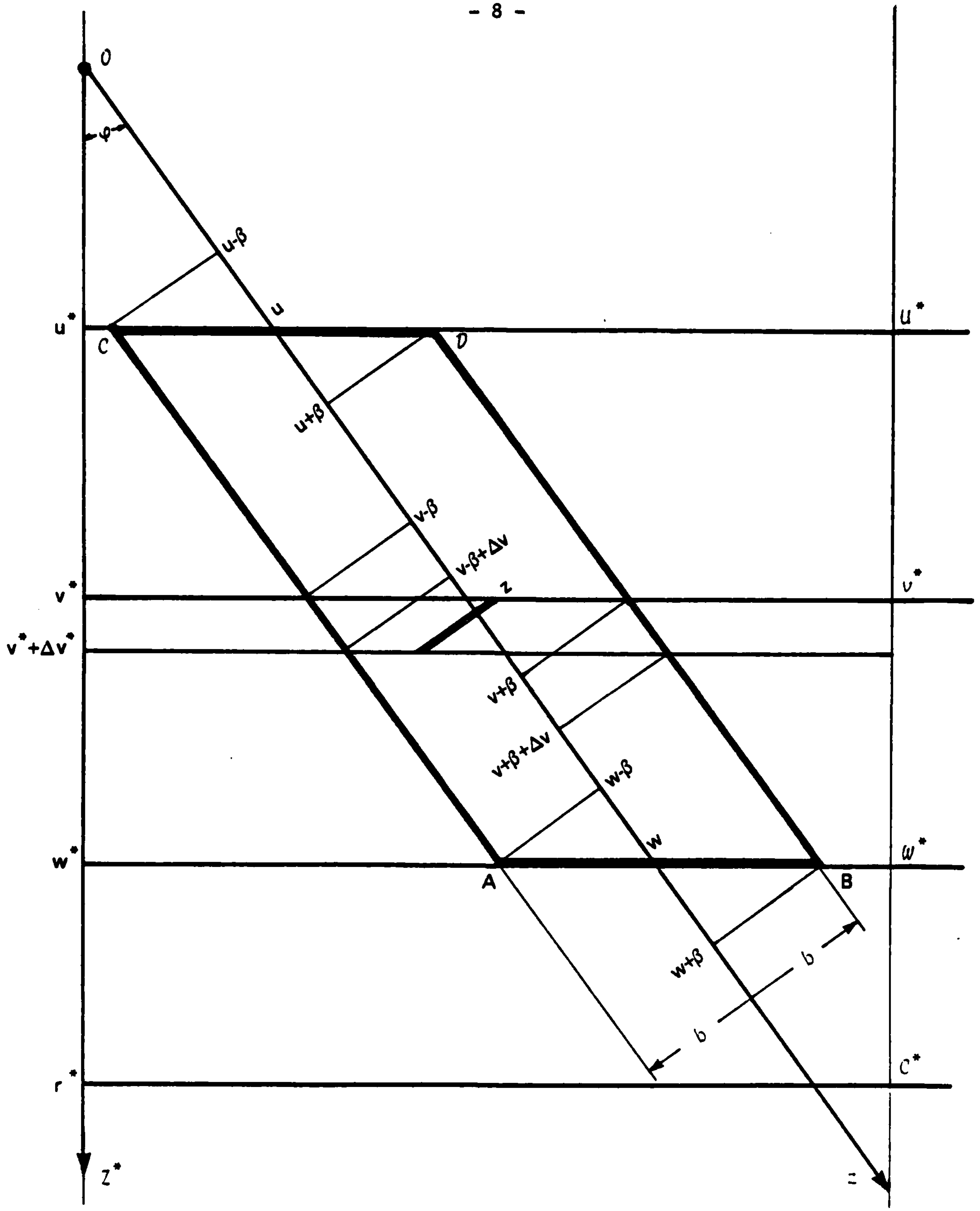


Fig. 2. The Geometry of the Reduction to A Two Dimensional Model

$$G(y) = \frac{1}{b} \int_0^b F((y^2+u^2)^{1/2})\, du \quad . \tag{3.2}$$

In the special case that X has a uniform distribution on [0,b] we obtain

$$\begin{aligned} G(y) = 1 - \frac{1}{2} (1-(\frac{y}{b})^2)^{1/2} - \frac{1}{2} (\frac{y}{b})^2 \log (\frac{y}{b}) \\ + \frac{1}{2} (\frac{y}{b})^2 \log (1 + (1-(\frac{y}{b})^2)^{1/2}) \quad , \quad 0 \le y \le b \quad . \end{aligned} \tag{3.3}$$

Notice that in this uniform case $G(0) = 1/2$ . It is expected therefore that half of the spheres in $P$ will not intersect $M$ . The case where the Poisson process of the disks centeres is homogeneous, and the distribution of Y is independent of the location of the disks, was called in [ 4 ] the standard case.

### 3.2 Random Disks on $M$ For $\beta > 0$

The projection of centers of random spheres on $M$ , in the case of $\beta > 0$ , yields centers of disks which follow a non-homogeneous Poisson process. Indeed, the volume of cylinder sets in the prism $P$ , corresponding to Borel sets of area A between the two parallel boundary lines on $M$ (of distances $u-\beta$ and $w+\beta$ from 0) is not constant. We can, however, partition the prism $P$ to a collection of thin layers, so that the centers of disks generated by spheres centered in each one of those layers practically follow a homogeneous Poisson process. More specifically, consider a layer bounded between two parallel planes $V^*$ and $V^{**}$ , having distances $v^*$ and $v^* + \Delta v^*$ from 0 , where

$$u^* \le v^* < v^* + \Delta v^* \le w^* \tag{3.4}$$

(see Fig. 2). These planes intersect $M$ in two parallel lines, with distances from 0 of $v$ and $v+\Delta v$ , respectively. We choose $\Delta v^*$ so that $\Delta v < \beta$ . The relationship between these parameters is

$$\frac{v^*}{v} = \frac{v^* + \Delta v^*}{v + \Delta v} = \frac{r^*}{r} = \cos \phi \quad . \tag{3.5}$$

Moreover, (3.4) and (3.5) imply the inequalities

$$u \le v < v + \Delta v \le w \quad ,$$

and (3.6)

$$\Delta v^* < \beta \cos \phi \quad .$$

Spheres centered within this layer generate on $M$ disks with centers between two parallel lines, whose distances from 0 are $v-\beta$ and $v+\Delta v+\beta$ . Furthermore, centers of disks which fall between the parallel lines of distances $v+\beta$ and $v+\Delta v-\beta$ follow a homogeneous Poisson process. The intensity of the homogeneous Poisson process for each such layer is $\mu \Delta v$ , where

$$\mu = \lambda \frac{b}{\beta} \quad . \tag{3.7}$$

Consider a sphere in the prism $P$ , centered on the plane $V^*$ in distance d from $M$ . This sphere generates on $M$ a disk whose center falls on a line of distance $z$ from 0 (see Fig. 2). Moreover,

$$|z-v| = d\beta/b \quad . \tag{3.8}$$

Accordingly, if $x$ is the radius of the sphere, the radius of the disk is

$$y = \left((x^2 - (z-v)^2 b^2/\beta^2)_+\right)^{1/2} \tag{3.9}$$

Let $G(y|z,v)$ denote the conditional c.d.f. of the random radius of a disk, $Y$ , given that its center is on a line having distance $z$ from 0 and the center of its generating sphere is on a plane intersecting $M$ at a distance $v$ from 0 . According to (3.9), this conditional c.d.f. is given by

$$G(y|z,v) = F\left((y^2 + (z-v)^2 b^2/\beta^2)^{1/2}\right) \quad . \tag{3.10}$$

It is interesting to note that

$$G(0|z,v) = F\left(\frac{b}{\beta}\,|z-v|\right) \quad . \tag{3.11}$$

Furthermore, $G(y|z,v) = 1$ for all $y \ge b(1 - (z-v)^2/\beta^2)^{1/2}$ .

## 4. Moments of The Visibility Measure on $C$

In the present section we define a measure of visibility on an interval $\overline{C}$ on $C$ and its moments. This definition and the derivation of the visibility moments on $\overline{C}$ follow the general methodology of [ 4 ], which was developed for the two-dimensional case. Since the curve $C$ and the boundaries $U$ and $W$ are parallel lines, it is more convenient to formulate the various formulae in terms of cartesian coordinates, rather than using the general formulae of [ 4 ], which are based on polar coordinates. In Fig. 4, we present the geometry of the shadowing process on $M$. The source of light, 0 , is the origin of the coordinate system. The x-axis is parallel to the line $C$. The centers of the disks are distributed in a [illegible]ip, $S$, between two parallel lines, whose $Z$-coordinates are $u-\beta$ and [illegible] , respectively. The interval of interest, $\overline{C}$, is bounded by the end poin[illegible] $(s',r)$ and $(s'',r)$. A point $P_s=(s,r)$ in $\overline{C}$ is called visible, i[illegible] [illegible]he line segment $\overline{OP}$ does not intersect any shadowing disk. The measure o[illegible] [illegible]sibility, $V\{C\}$, is the total portion of segments of $\overline{C}$ which are visible. Let

$$I(x) = \begin{cases} 1 & \text{, if } P_x = (x,r) \text{ is visible} \\ 0 & \text{, otherwise} \end{cases} \quad , \tag{4.1}$$

then

$$V\{\overline{C}\} = \int_{s'}^{s''} I(x)dx \quad . \tag{4.2}$$

If $p(s)$ is the probability that the point $P_s = (s,r)$ is visible, then

$$E\left\{V\{\overline{C}\}\right\} = \int_{s'}^{s''} p(s)ds \quad . \tag{4.3}$$

Similarly, if $s_1,\ldots,s_n$ are the x-coordinates of $n$ points in $\overline{C}$, and $p(s_1,\ldots,s_n)$ is the probability that these $n$ points are simultaneously visible, then, as shown in [ 4 ], the n-th moment of $V\{\overline{C}\}$ is given by

$$E\left\{V^n\{\overline{C}\}\right\} = n! \int \underset{A}{\cdots} \int p(s_1,\ldots,s_n)\, ds_1 \ldots ds_n \tag{4.4}$$

where $A = \{s' \le s_1 < \ldots < s_n \le s''\}$ .

The visibility probabilities $p(s_1,\ldots,s_n)$ , for all $n\geq 1$ , are given, as in [ 4] , by employing the functions defined below.

Let $\lambda K_-(s,t)$ be the expected number of disks falling within the intersection of the triangle $OP_{s-t}P_s$ and the strip $S$ , which do not intersect the line $\overline{OP_s}$ . (We include also disks with zero radius). Similarly, we define the function $\lambda K_+(s,t)$ , which corresponds to the triangle $OP_sP_{s+t}$ (see Fig. 4). These functions will be explicitly developed for the standard ($\beta=0$) and non-standard ($\beta>0$) cases in the following sections. It will be shown that $K_-(s,t) = K_+(s,t) = K(s,t)$ . Let $C$ be a subset of the strip $S$ that contains the centers of all disks which may cast shadows on $\overline{C}$ (see Fig. 3). This set is obtained by the intersection of $S$ with the triangle $OP_{\theta'}P_{\theta''}$ , where

$$\theta' \leq s' - b\left((s')^2 + r^2\right)^{1/2}/(u-\beta) \tag{4.5}$$

and

$$\theta'' \geq s'' + b\left((s'')^2 + r^2\right)^{1/2}/(u-\beta) \tag{4.6}$$

Let $\nu\{C\}$ be the expected number of disks falling in $C$ (including ones having a zero radius). Explicit formulae for the standard and non-standard cases will be given later. Accordingly, the expected number of disks that may cast shadow on a point $P_s$ in $\overline{C}$ is $\nu\{C\} - \lambda(K(s,s-\theta') + K(s,\theta''-s))$ . From the Poisson field assumptions we obtain that the probability that $P_s$ is visible is

$$p(s) = \exp\left\{-\left[\nu\{C\} - \lambda\left(K(s,s-\theta') + K(s,\theta''-s)\right)\right]\right\} \tag{4.7}$$

For any $n$ points $P_{s_1},\ldots,P_{s_n}$ in $\overline{C}$ , such that $s'\leq s_1<\ldots<s_n\leq s''$ , the probability of their simultaneous visibility is

$$\begin{aligned} p(s_1,\ldots,s_n) = \exp\Big\{-\Big[\nu\{C\} - \lambda\Big(K(s_1,s_1-\theta') + \\ K(s_n,\theta''-s_n)\Big) - \lambda\sum_{i=1}^{n-1}\left(K(s_i,\tilde{s}_i-s_i) + K(s_{i+1},s_{i+1}-\tilde{s}_i)\right)\Big]\Big\} \end{aligned} \tag{4.8}$$

where

$$\tilde{s}_i = r\tan\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{s_i}{r}\right) + \tan^{-1}\left(\frac{s_{i+1}}{r}\right)\right)\right) \tag{4.9}$$

The point $P_{\tilde{s}_i}$ is chosen on the bisector of the angle between $\overline{OP}_{s_i}$ and $\overline{OP}_{s_{i+1}}$ in order to assure that a disk whose center is in the sector between $\overline{OP}_{s_i}$ and $\overline{OP}_{\tilde{s}_i}$, which does not intersect $\overline{OP}_{s_i}$, will not intersect $\overline{OP}_{s_{i+1}}$, and vice versa.

The probability that the whole interval $\overline{C}$ is visible is

$$P_1 = \exp -[\nu^*\{C\} - \lambda\ K(s',\theta'-\theta') + K(s'',\theta''-s'')\ ] \quad , \tag{4.10}$$

where $\nu^*\{C\}$ is the expected number of disks falling in $S$ between $\overline{OP}_{\theta'}$ and $\overline{OP}_{\theta''}$, minus the expected number of disks falling in $S$, between $\overline{OP}_{s'}$ and $\overline{OP}_{s''}$ having zero radius.
Finally, as shown in [ 3] and [ 4 ] , the moments of $V\{\overline{C}\}$ can be determined recursively, in the following manner. Let

$$\Psi_0(s) = \exp\{\lambda K(s,s-\theta')\} \quad , \quad s'\le s\le s'' \tag{4.11}$$

and, for every $j\ge 1$, define recursively

$$\Psi_j(s) = \int_{s'}^{s} \Psi_{j-1}(y)\ \exp\{\lambda[K(y,\tilde{y}_1(s)) + K(s,\tilde{y}_2(s))]\}\ dy \quad , \tag{4.12}$$

where

$$\tilde{y}_1(s) = r\ \tan\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{y}{r}\right) + \tan^{-1}\left(\frac{s}{r}\right)\right)\right) - y \tag{4.13}$$

and

$$\tilde{y}_2(s) = s-y-\tilde{y}_1(s) \quad . \tag{4.14}$$

The n-th moment of $V\{\overline{C}\}$ is given then by the formula

$$\mu_n = n!\ \exp\{-\nu\{C\}\} \int_{s'}^{s''} \Psi_{n-1}(s)\ \exp\{\lambda K(s,\theta''-s)\}\ ds \quad . \tag{4.15}$$

In the following section we develop more explicit formulae for these functions.

## 5. More Explicit Formulae for $K(s,t)$ and $\nu\{C\}$

### 5.1 The Standard Case ($\beta=0$)

In the standard case, the expected number of disks in $C$ is the intensity, $2\lambda b$, multiplied by the area of $C$, i.e.,

$$\nu\{C\} = \lambda b \, \frac{\theta''-\theta'}{r} (w^2-u^2) \quad ,$$

and (5.1)

$$\nu^*\{C\} = \nu\{C\} - \frac{\lambda\bar{b}}{2r} (w^2-u^2)(s''-s') \quad .$$

The geometry of the standard case, (5.1) as illustrated in Fig. 3, shows that the location of centers of disks of radius $y$, which are between the rays $\overline{OP}_{s-t}$ and $\overline{OP}_s$, and which do not intersect the ray $\overline{OP}_s$, is the trapezoid $E_yF_yC_yD_y$. Similarly, the trapezoid $E'_y F'_y \underline{C'_y} D'_y$ contains the centers of all disks of radius $y$, which are between $\overline{OP}_s$ and $\overline{OP}_{s+t}$, which do not intersect $\overline{OP}_s$. Notice that the area of $E_yF_yC_yD_y$ is the same as that of $E'_yF'_yC'_yD'_y$. Finally, since $K_-(s,t)$ and $K_+(s,t)$ are the average areas of these trapezoids, with respect to the distribution $G(y)$, multiplied by $2b$, we obtain that

$$K_-(s,t) = K_+(s,t) \equiv K(s,t) \quad , \quad \text{for all} \quad s'\le s\le s'' \tag{5.2}$$

and all $0 < t$.

We develop now the formula for $K(s,t)$. Notice first that the area of the trapezoid $E_yF_yC_yD_y$ can be computed as the area of the triangle $B_yC_yD_y$ minus the area of the triangle $B_yE_yF_y$. Let $A(s,t,v,y)$ be the area of such a triangle, whose vertices are $B_yC''_yD''_y$, where $C''_y$ and $D''_y$ are on a line parallel to the x-axis, at distance $v$ from the origin. The triangle $\underline{B_yC''_y}D''_y$ is similar to the triangle $OP_{s-t}P_s$, since $\overline{B_yD'_y} \,||\, \overline{OP}_s$. Hence, $\overline{C''_yD''_y}+d = \frac{w}{r}\,t$, and the area of the triangle $OC''_yA''$ is $\frac{v^2t}{2r}$. The area of the parallelogram $B_yA'D''_yA''$ is $d(v-h)$. The area of the triangle $OB_yA'$ is $dh/2$. Moreover, $h = \frac{r}{t}\,d$, and since the triangles $AP_{s-d}P_s$ and $OP_{s}P_s$ are similar, $d = y(s^2+r^2)^{1/2}/r$. From all these geometrical considerations we obtain that,

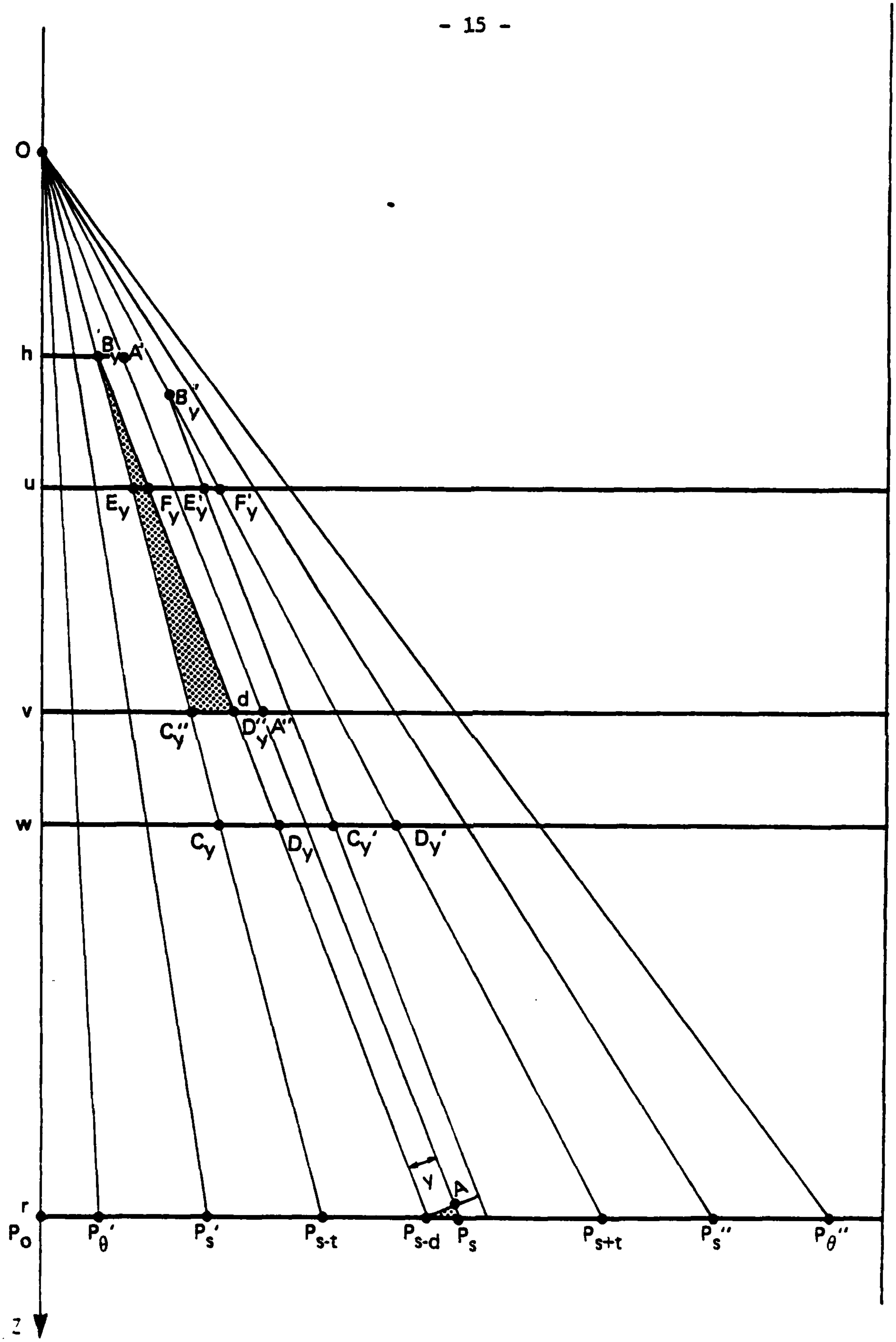


Fig. 3. The Geometry on $M$: Standard Case.

$$A(s,t,v,y) = \begin{cases} (vt/(s^2+r^2)^{1/2}-y)^2 \dfrac{s^2+r^2}{2tr} & , \quad \text{if} \quad y \le vt/(s^2+r^2)^{1/2} \\ 0 & , \quad \text{otherwise} \end{cases} \tag{5.3}$$

Finally, let

$$\hat{K}(s,t,v,r) = b\,\frac{s^2+r^2}{tr} \int_0^{\min(b,vt/(s^2+r^2)^{1/2})} \left(y - vt/(s^2+r^2)^{1/2}\right)^2 dG(y) \quad , \tag{5.4}$$

then $K(s,t) = \hat{K}(s,t,w,r) - \hat{K}(s,t,u,r)$ .

In Section 6 we further develop this formula, for the uniform distribution $F(x)$ . In this special case $G(y)$ is given by formula (3.3).

### 5.2 The Non-Standard Case ($\beta>0$)

The expected number of disks centered in $C$ is, in the non-standard case, the same as in the standard case. Indeed,

$$\begin{aligned} \nu\{C\} &= \frac{\lambda b}{\beta} \cdot \frac{\theta''-\theta'}{r} \int_u^w dv \int_{v-\beta}^{v+\beta} z\,dz \\ &= \frac{\lambda b}{r} (\theta''-\theta')(w^2-u^2) \quad . \end{aligned} \tag{5.5}$$

In a similar fashion we obtain that

$$\begin{aligned} \nu^*\{C\} &= \nu\{C\} - \frac{\lambda b}{\beta} \cdot \frac{s''-s'}{r} \int_u^w dv \int_{v-\beta}^{v+\beta} z\,F\!\left(b\frac{|z-v|}{\beta}\right) dz \\ &= \nu\{C\} - \frac{\lambda b}{r} (s''-s') \int_u^w dv \int_{-1}^{1} (u+v\beta)\,F(b|u|)\,du \quad . \end{aligned}$$

In the special case of uniform distribution

$$F(x) = \frac{1}{b}\min(x,b) \ , \ 0 \le x \quad , \quad \text{one gets,}$$

$$\nu^*\{C\} = \nu\{C\} - \frac{\lambda b}{2r} (s''-s')(w^2-u^2) \quad .$$

The K-functions are obtained in the following manner. As seen in Fig. 4, consider a disk with a center on a parallel line, $Z$ , intersecting the axis at $z$ , $u-\beta\le z\le w+\beta$ , and having distance $x$ from the intersection of $Z$ with $\overline{OP_s}$ . The maximal radius that such a disk can have, without inter secting $\overline{OP_s}$ , is

$$y(x) = xr/(s^2+r^2)^{1/2} \quad , \quad 0\le x\le \frac{z}{r}\, t \quad . \tag{5.6}$$

Since this structure is the same for disks on the right or the left sides of $\overline{OP_s}$ , $K_-(s,t) = K_+(s,t) \equiv K(s,t)$ , where

$$K(s,t) = \frac{b}{\beta} \int_u^w dv \int_{v-\beta}^{v+\beta} dz \int_0^{zt/r} G\,(xr/(s^2+r^2)^{1/2} \mid z,v)\, dx \quad . \tag{5.7}$$

Substituting (3.10) in (5.6) and making simple change of variables, one obtains

$$K(s,t) = \frac{b}{r}\,(s^2+r^2)^{1/2} \int_u^w dv \int_{-1}^{1} dz \int_0^{(v+\beta z)t/(s^2+r^2)^{1/2}} F((y^2+b^2z^2)^{1/2})\, dy \quad . \tag{5.7}$$

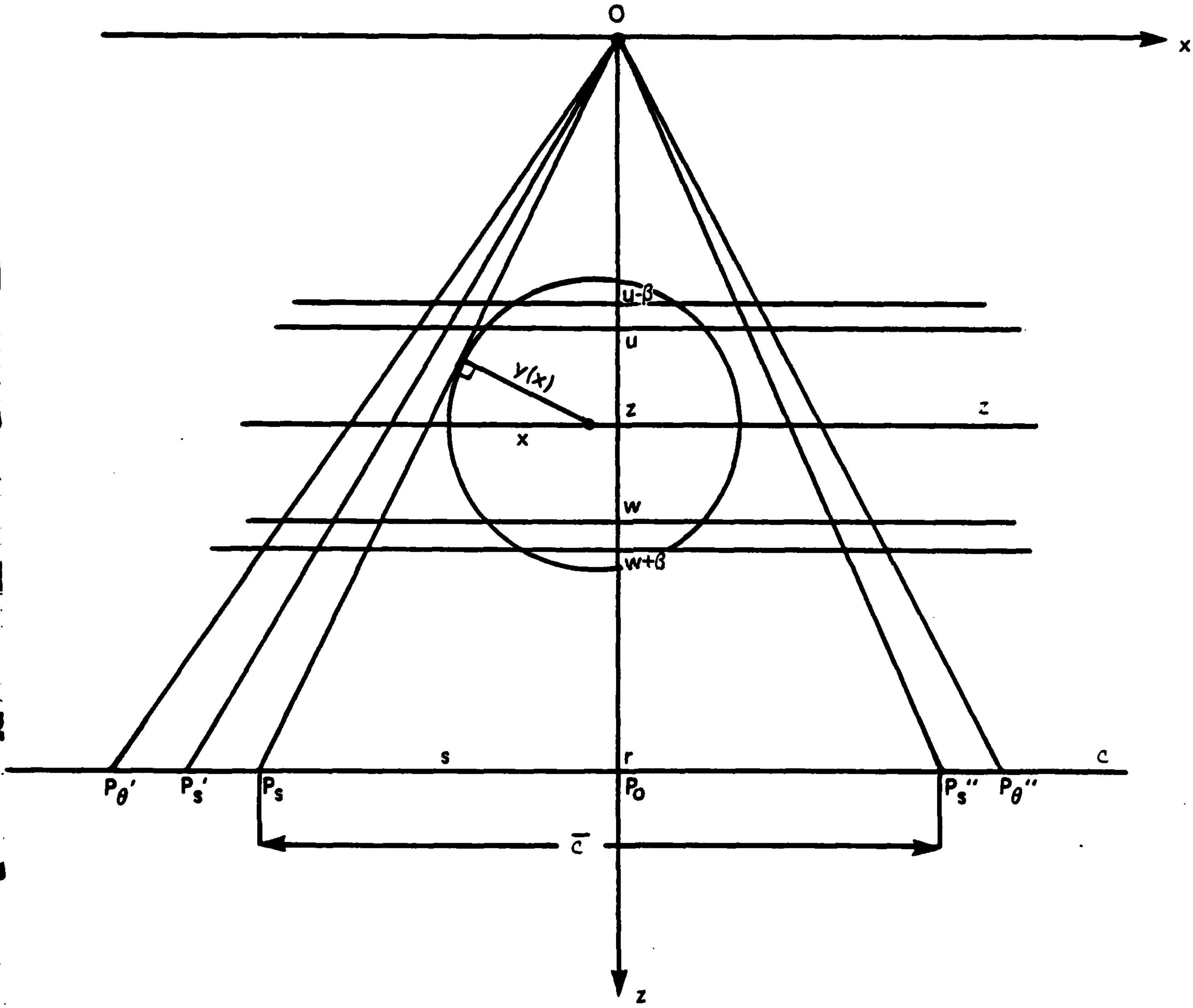


Fig. 4. The Geometry On $\mathcal{M}$: Non-Standard Case.

## 6. Further Development for the Uniform Case and Numerical Illustrations

### 6.1 The Standard Case

In the present section we provide more explicit expression for the function $\hat{K}(s,t,v,r)$ , given in (5.3), when the distribution of the radii of spheres, $F(x)$ , is uniform on $(0,b)$ , i.e. $F(x) = \frac{1}{b}\min(x,b)$ , for all $0 \le x < \infty$ . The corresponding formula of $G(y)$ , for the standard case if given in (3.3).

We distinguish between two cases: Case I, the case of $b \le vt/\left(s^2+r^2\right)^{1/2}$ and Case II, the case of $b > vt/(s^2+r^2)^{1/2}$ .

Case I

According to (5.3) ,

$$\hat{K}(s,t,v,r) = \frac{s^2+r^2}{tr}\, b \left[ E\{Y^2\} - 2\,\frac{vt}{\left(s^2+r^2\right)^{1/2}}\, E\{Y\} + \frac{v^2t^2}{s^2+r^2} \right] . \tag{6.1}$$

Formulae for the first two moments of $Y$ in the standard case are developed in Appendix A.1. According to these formulae

$$\hat{K}(s,t,v,r) = .291667b^3\,\frac{s^2+r^2}{tr} - .745922b^2\, v(s^2+r^2)^{1/2}/r + bt\, v^2/r . \tag{6.2}$$

Case II

In Case II,

$$\hat{K}(s,t,v,r) = \frac{s^2+r^2}{tr}\, b \int_0^{\xi} (y-\xi)^2 d\, G(y) , \tag{6.3}$$

where

$$\xi = vt/\left(s^2+r^2\right)^{1/2} .$$

Equivalently,

$$\hat{K}(s,t,v,r) = bv^2t/r - 2\frac{bv}{r}\, s^2+r^2\ ^{1/2} \cdot \int_0^{\xi} y\, dG(y) + \frac{s^2+r^2}{tr}\, b \int_0^{\xi} y^2\, dG(y) . \tag{6.4}$$

In Appendix II we evaluate the incomplete moments $E\{Y^i;\xi\} = \int_0^\xi y^i dG(y)$ , $i=1,2$ .

According to (6.4) and formulae (A.12) and (A.19) we obtain

$$\hat{K}(s,t,v,r) = bv^2t/r - \frac{b^2v}{r}(s^2+r^2)^{1/2} \cdot \left\{ \frac{1}{2}\tan^{-1}\left(\frac{(\xi/b)^2}{1-(\xi/b)^2}\right)^{1/2} - \frac{2}{3}(\frac{\xi}{b})^3\left(\log(\frac{\xi}{b})+\frac{1}{6}\right) - \frac{1}{2}A(\frac{\xi}{b}) - \frac{1}{2}(\frac{\xi}{b})\left(1-(\frac{\xi}{b})^2\right)^{1/2} + (\frac{\xi}{b})^3\log\left(1+(1-(\frac{\xi}{b})^2)^{1/2}\right)\right\} + \frac{s^2+r^2}{2tr}b^3\left\{\frac{1}{3} - \frac{1}{3}\left(1-(\frac{\xi}{b})^2\right)^{1/2}\left(1+\frac{1}{2}(\frac{\xi}{b})^2\right) - \frac{1}{2}(\frac{\xi}{b})^4\log\left((\xi b)/(1+(1-(\frac{\xi}{b})^2)^{1/2})\right)\right\}, \tag{6.5}$$

where $A(z) \cong .0933z^2 + .57z^4 - .3633z^6$ .

## 6.2 The Non-Standard Case ($\beta>0$)

The function $K(s,t)$ in the non-standard case is evaluated, for the uniform distribution of sphere radii, but substituting $F(x) = \frac{1}{b}\min(x,b)$ , $0\le x<\infty$ , in formula (5.7). Since

$$F\,(y^2+b^2z^2)^{1/2} = \begin{cases} ((\frac{y}{b})^2+z^2)^{1/2} & , \quad \text{if } y \le b(1-z^2)^{1/2} \\ 1 & , \quad \text{if } y > b(1-z^2)^{1/2} \end{cases}$$

we distinguish between two cases:

<u>Case I</u> $(v+z\beta)t/b(s^2+r^2)^{1/2} \le (1-z^2)^{1/2}$ .

Let $\gamma(t) = t/(s^2+r^2)^{1/2}$ and $g(z,v,t) = \gamma(t)(v+z\beta)/b$ .

Then, in Case I,

$$\int_0^{bg(z,v,b)} F\left((y^2+b^2z^2)^{1/2}\right)dy = \frac{b}{2}\left[g(z,v,t)\,(g^2(z,v,t)+z^2)^{1/2} + z^2\log(g(z,v,t) + g^2(z,v,t)+z^2)^{1/2}) - z^2\log|z|\right] . \tag{6.7}$$

<u>Case II</u> $(v + z\beta)t/b(s^2+r^2)^{1/2} > (1-z^2)^{1/2}$

In Case II,

$$\int_0^{b\tilde{g}(z,v,t)} F((y^2+b^2z^2)^{1/2})dy = \frac{b}{2}\Big[z^2\log(1+(1-z^2)^{1/2})$$

$$- z^2\log|z| + 2g(z,v,t) - (1-z^2)^{1/2}\Big] \quad . \tag{6.8}$$

Thus, from (5.7), (6.7) and (6.8) ,

$$K(s,t) = \frac{b^2}{2}\,\frac{(s^2+r^2)^{1/2}}{r}\int_u^w dv \int_{-1}^{1} dz \Big[I\{g(z,v,t) \le (1-z^2)^{1/2}\} \cdot$$

$$\Big(g(z,v,t)(g^2(z,v,t) + z^2\Big)^{1/2} + z^2\log\Big(g(z,v,t) + \Big(g^2(z,v,t) + z^2\Big)^{1/2}\Big)$$

$$-z^2\log|z|\Big) + I\{g(z,v,t) > (1-z^2)^{1/2}\}\ \Big(z^2\log(1+(1-z^2)^{1/2}\Big)$$

$$-z^2\log|z| + 2\gamma(z,v,t) - (1-z^2)^{1/2}\Big)\Big] \quad , \tag{6.9}$$

where $I\{A\}$ designates the indicator function of the set $A$ . The double integral in (6.9) was evaluated numerically, applying an iterative procedure with a 20-point Gaussian quadrature (see Abramovitz and Stegun [1;pp. 916]).

## 6.3 Numerical Illustrations

In the present section we provide some numerical computations, which illustrate the above theory. The following parameters are held fixed throughout the various cases.

1. The distance of $\mathcal{U}^*$ from the origin is $u^* = .4$ .
2. The distance of $\mathcal{W}^*$ from the origin is $w^* = .6$ .
3. The distance of $C^*$ from the origin is $r^* = .1$ .

The radii of the spheres are i.i.d. random variables having a uniform distribution on $[0,b]$ , with $b = .3$ .

The line segment $\overline{\mathcal{C}}$ is of 2 units length, and is located between $s' = -1.$ and $s'' = 1.$ The varying parameters are: the intensity of the Poisson field, $\lambda = 2,6,10$ and the inclination angle of the plane $M$ , $\phi = 0$ , $\pi/8$ , $\pi/4$ and $\pi/3$ . In Table 6.1 we present, for each one of these twelve cases, the first ten standardized moments of $V\{\overline{\mathcal{C}}\}$ and the moments of the approximating mixed-beta distribution. The standardized moments of $V\{\overline{\mathcal{C}}\}$ are $\mu_n^* = \mu_n/(s''-s')^n$ , $n = 1,2,\ldots$ , where $\mu_n$ have

been determined according to the recursive formula (4.19). The mixed-beta distribution is a distribution function on [0,1] , having a c.d.f.

$$F^*(x) = \begin{cases} P_0 + (1 - P_0 - P_1) \frac{1}{B(\alpha,\beta)} \int_0^x u^{\alpha-1} (1-u)^{\beta-1} du & , \quad 0 \le x < 1 \\ 1 & , \quad 1 \le x \end{cases} \qquad (6.10)$$

This is a beta-distribution on $(0,1)$ , mixed with a two-point distribution, with jumps of height $P_0$ at $x=0$ and $P_1$ at $x=1$ .

$$P_0 = P\{V\{\overline{C}\} = 0\} \text{ and } P_1 = P\{V\{\overline{C}\} = s''-s'\} \quad .$$

$P_1$ is given by formula (4.14). In [3] and [4] we have shown how the parameters $P_0$, $\alpha$ and $\beta$ can be determined, by equating the first three standardized moments of $V\{\overline{C}\}$ , $\mu_n^*$ , $n=1,2,3$ , to the first three moments of $F^*(x)$ . In Table 6.2 we present the parameters of the approximating mixed-beta distribution, $F^*(x)$ , for the twelve cases under consideration. The method of computing the convolutions in the iterative formulae of the moments was described in [4]. To apply those formulae to the present cases we substitute, $r=r^*/\cos\phi$ , $w=w^*/\cos\phi$ , $\mu=\mu^*/\cos\phi$ and $\beta=b\tan\phi$ , in the proper places. We see in Table 6.1 and Table 6.2 that for each intensity $\lambda$ , the total visibility probabilities, $P_1$ , as well as the moments of $V\{\overline{C}\}$ decrease as the inclination angle $\phi$ increases.

Table 6.1 shows again very good agreement between the first ten standardized moments of $V\{\overline{C}\}$ and those of the approximating beta-mixture distribution. Thus, one can apply the beta-mixture distribution to approximate the c.d.f. of $V\{\overline{C}\}/(s''-s')$ . Thus,

$$P\{V\{\overline{C}\} \le v(s''-s')\} \cong \begin{cases} P_0 + (1-P_0-P_1) I_v(\alpha,\beta) & , \quad 0 \le v < 1 \\ 1 & , \quad v \ge 1 \end{cases} ,$$

where $I_v(\alpha,\beta)$ is the incomplete beta function ratio.

Table 6.1 Standardized Moments of Visibility and Their Mixed-Beta Approximations*), For $s'=-1$ , $s''=1$ , $\phi=0$ , $\pi/8$ , $\pi/4$ , $3\pi/8$ and $\lambda=2,6,10$ , $r=1$ , $w=.6$ , $u=.4$ , $b=.3$

| $\lambda$ | $\phi$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | .958 | .932 | .914 | .902 | .894 | .889 | .886 | .886 | .887 | .887 | .841 |
| | | .958 | .932 | .914 | .902 | .893 | .886 | .881 | .876 | .873 | .870 | .841 |
| | $\pi/8$ | .955 | .927 | .909 | .896 | .888 | .882 | .879 | .878 | .879 | .882 | .831 |
| | | .955 | .927 | .909 | .896 | .886 | .879 | .873 | .868 | .865 | .861 | .831 |
| | $\pi/4$ | .944 | .910 | .887 | .871 | .860 | .853 | .848 | .846 | .845 | .848 | .791 |
| | | .944 | .910 | .887 | .871 | .859 | .850 | .843 | .837 | .832 | .828 | .791 |
| | $3\pi/8$ | .904 | .847 | .809 | .783 | .765 | .752 | .743 | .737 | .733 | .732 | .678 |
| | | .904 | .847 | .809 | .783 | .764 | .749 | .783 | .729 | .721 | .715 | .678 |
| 6 | 0 | .878 | .809 | .765 | .735 | .713 | .698 | .688 | .680 | .675 | .674 | .594 |
| | | .878 | .809 | .765 | .734 | .712 | .696 | .683 | .672 | .664 | .654 | .594 |
| | $\pi/8$ | .871 | .798 | .751 | .720 | .697 | .681 | .670 | .662 | .657 | .655 | .573 |
| | | .871 | .798 | .751 | .719 | .696 | .679 | .665 | .654 | .646 | .639 | .573 |
| | $\pi/4$ | .842 | .754 | .699 | .661 | .635 | .616 | .602 | .593 | .586 | .582 | .494 |
| | | .842 | .754 | .699 | .661 | .634 | .614 | .598 | .586 | .576 | .567 | .494 |
| | $3\pi/8$ | .739 | .609 | .532 | .482 | .448 | .423 | .406 | .392 | .382 | .375 | .284 |
| | | .739 | .609 | .532 | .482 | .447 | .422 | .402 | .387 | .375 | .365 | .284 |
| 10 | 0 | .806 | .703 | .640 | .599 | .570 | .549 | .534 | .523 | .515 | .510 | .420 |
| | | .806 | .703 | .640 | .599 | .569 | .547 | .530 | .517 | .506 | .497 | .420 |
| | $\pi/8$ | .794 | .687 | .622 | .579 | .549 | .527 | .511 | .500 | .492 | .486 | .396 |
| | | .794 | .687 | .622 | .578 | .548 | .525 | .508 | .494 | .483 | .474 | .396 |
| | $\pi/4$ | .750 | .626 | .551 | .503 | .470 | .446 | .429 | .416 | .407 | .400 | .309 |
| | | .750 | .626 | .551 | .503 | .469 | .444 | .425 | .411 | .399 | .389 | .309 |
| | $3\pi/8$ | .603 | .440 | .352 | .299 | .264 | .240 | .223 | .210 | .200 | .193 | .123 |
| | | .603 | .440 | .352 | .299 | .263 | .238 | .220 | .206 | .195 | .987 | .123 |

*) The upper value in each cell is the standardized moment of $V\{\tilde{C}\}$ . The lower value is that of the beta-mixture distribution. $\mu_\infty^*=P_1$ .

Table 6.2 The Parameters of The Mixed-Beta Distributions, For $s'=1$ , $s''=1$ , $\phi=0$ , $\pi/8$ , $\pi/4$ , $3\pi/8$ and $\lambda=2,6,10$ , $r=1.$ , $w=.6$ , $u=.4$ , $b=.3$

| $\lambda$ | $\phi$ | $P_0$ | $P_1$ | $\alpha$ | $\beta$ | $\sigma$ |
|---|---|---|---|---|---|---|
| 2 | 0 | 0 | .8409 | 3.489 | 1.293 | .12029 |
| | $\pi/8$ | 0 | .8307 | 3.525 | 1.301 | .12348 |
| | $\pi/4$ | 0 | .7905 | 3.595 | 1.324 | .13578 |
| | $3\pi/8$ | .0001 | .6576 | .3474 | 1.354 | .17217 |
| 6 | 0 | .0007 | .5945 | 3.132 | 1.336 | .19392 |
| | $\pi/8$ | .0010 | .5733 | 3.114 | 1.339 | .19814 |
| | $\pi/4$ | .0021 | .4940 | 2.999 | 1.350 | .21371 |
| | $3\pi/8$ | .0067 | .2844 | 2.474 | 1.387 | .25194 |
| 10 | 0 | .0039 | .4204 | 2.724 | 1.348 | .23307 |
| | $\pi/8$ | .0045 | .3956 | 2.683 | 1.352 | .23705 |
| | $\pi/4$ | .0068 | .3088 | 2.492 | 1.373 | .25082 |
| | $3\pi/8$ | .0159 | .1230 | 1.872 | 1.483 | .27510 |

## References

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## Appendix I. The Moments of Y in the Standard Uniform Case

Writing $E\{Y\} = \int_0^b (1-G(y))dy$ and substituting (3.3) we obtain

$$E\{Y\} = \frac{1}{2}\int_0^b (1-(\frac{y}{b})^2)^{1/2}dy + \frac{1}{2}\int_0^b (\frac{y}{b})^2\log(\frac{y}{b})dy$$

$$-\frac{1}{2}\int_0^b (\frac{y}{b})^2\log(1+\sqrt{1-(\frac{y}{b})^2})\,dy \quad . \qquad (A.1)$$

Making the changes of variable, $u=y/b$ , we obtain

$$E\{Y\} = \frac{b}{2}\{\int_0^1 (1-u^2)^{1/2}du + \int_0^1 u^2 \log u\, du$$

$$-\int_0^1 u^2\log(1+\sqrt{1-y^2})du\} \quad . \qquad (A.2)$$

Furthermore, making the transformation $w=u^2$ , we obtain

$$\int_0^1 (1-u^2)^{1/2}du = \frac{1}{2}\int_0^1 w^{-1/2}(1-w)^{1/2}dw$$

$$= \frac{1}{2}B(\frac{1}{2},\frac{3}{2}) = \pi/4 \quad , \qquad (A.3)$$

where $B(a,b)$ designates the beta-function at $a$ and $b$ . In addition the other integrals of (A.2) are (see H.B. Dwight [2;pp. 141])

$$\int_0^1 u^2\log u\, du = -1/9 \quad ,$$

$$\int_0^1 u^2\log(1+\sqrt{1-u^2})du = .1505865 \ldots$$

Substituting these results in (A.2) we obtain the first moment

$$E\{Y\} = b\{\frac{\pi}{8} - \frac{1}{18} - \frac{1}{2}(.1505865)\} = .261850\, b \quad . \qquad (A.4)$$

The second moment of $Y$ is obtained in a similar fashion as

$$\begin{aligned} E\{Y^2\} &= 2\int_0^b y[1-G(y)]dy \\ &= b\int_0^b (\tfrac{y}{b})(1-(\tfrac{y}{b})^2)^{1/2}dy + b\int_0^b (\tfrac{y}{b})^3\log(\tfrac{y}{b})\,dy \\ &\quad - b\int_0^b (\tfrac{y}{b})^3\log(1+(1-(\tfrac{y}{b})^2)^{1/2})dy \\ &= .166667 \end{aligned} \tag{A.5}$$

Appendix II. The Incomplete Moments of $Y$ In The Standard Uniform Case

The first two incomplete moments of $Y$ are

$$E\{Y^i;\zeta\} = \int_0^\zeta y^i dG(y) \quad , \quad i=1,2 \text{ , where } \zeta<b \quad . \tag{A.6}$$

The incomplete moment of the first order is

$$E\{Y;\zeta\} = \int_0^\zeta [1-G(y)]dy - \zeta[1-G(\zeta)] \quad . \tag{A.7}$$

Substituting (3.3) for $G(y)$ we obtain that

$$\begin{aligned} \int_0^\zeta (1-G(y))dy &= \frac{1}{2}\int_0^\zeta (1-(\tfrac{y}{b})^2)^{1/2}dy + \frac{1}{2}\int_0^\zeta (\tfrac{y}{b})^2\log(\tfrac{y}{b})\,dy \\ &\quad - \frac{1}{2}\int_0^\zeta (\tfrac{y}{b})^2\log(1+\sqrt{1-(\tfrac{y}{b})^2})\,dy \quad . \end{aligned} \tag{A.8}$$

Using a result from Abramowitz and Stegun [1;pp. 946], we can show that

$$\frac{1}{2}\int_0^\zeta (1-(\tfrac{y}{b})^2)^{1/2}dy = \frac{b}{4}[\tan^{-1}(\frac{(\zeta/b)^2}{1-(\zeta/b)^2})^{1/2} + \frac{\zeta}{b}(1-(\tfrac{\zeta}{b})^2)^{1/2}] \quad . \tag{A.9}$$

Furthermore,

$$\frac{1}{2}\int_0^{\zeta} (\frac{y}{b})^2 \log(\frac{y}{b})\,dy = \frac{b}{6}(\frac{\zeta}{b})^3[\log(\frac{\zeta}{b})-\frac{1}{3}] \quad . \tag{A.10}$$

Finally, by making the transformation $w=(1-(y/b)^2)^{1/2}$ , we obtain

$$\int_0^{\zeta}(\frac{y}{b})^2\log(1 + (1-(y/b)^2)^{1/2})dy = \frac{b}{2}\int_0^{(\zeta/b)^2} w^{1/2}\log(1 + (1-w)^{1/2})dw$$

$$\cong \frac{b}{2}[.0933(\frac{\zeta}{b})^2 + .57(\frac{\zeta}{b})^4 - .3633(\frac{\zeta}{b})^6] \quad . \tag{A.11}$$

The approximation on the RHS of (A.11) was obtained by approximating $w^{1/2}\log(1 + (1-w)^{1/2})$ on the interval (0,1) by the 2nd degree polynomial $.0933 + 1.14w - 1.09w^2$ . From (A.7) - (A.11) and (3.3) one obtains,

$$\begin{aligned} E\{Y;\zeta\} = b\{&\frac{1}{4}\tan^{-1}(\frac{(\zeta/b)^2}{1-(\zeta/b)^2})^{1/2} - \frac{1}{3}(\frac{\zeta}{b})^3(\log(\frac{\zeta}{b}) + \frac{1}{6}) \\ &- \frac{1}{4}A(\frac{\zeta}{b}) - \frac{1}{4}(\frac{\zeta}{b})(1 - (\frac{\zeta}{b})^2)^{1/2} + \frac{1}{2}(\frac{\zeta}{b})^3\log(1 + (1 - (\frac{\zeta}{b})^2)^{1/2})\} \quad , \end{aligned} \tag{A.12}$$

where

$$A(z) = .0933z^2 + .57z^4 - .3633z^6 \quad . \tag{A.13}$$

The incomplete moment of the second order is

$$E\{Y^2;\zeta\} = 2\int_0^{\zeta} y[1-G(y)]dy - \zeta^2(1-G(\zeta)) \quad . \tag{A.14}$$

From (3.3) we obtain

$$\begin{aligned} 2\int_0^{\zeta} y[1-G(y)]dy = &\int_0^{\zeta} y(1-(\frac{y}{b})^2)^{1/2}dy - b\int_0^{\zeta} (\frac{y}{b})^3\log(\frac{y}{b})dy \\ &- \int_0^{\zeta} (\frac{y}{b})^3\log(1 + (1-(\frac{y}{b})^2)^{1/2})dy \end{aligned} \tag{A.15}$$

Following the integration methods applied above, we obtain

$$\int_0^\zeta y(1-(\frac{y}{b})^2)^{1/2}dy = \frac{b^2}{3}[1-(1-(\frac{\zeta}{b})^2)^{3/2}] \quad , \tag{A.16}$$

and

$$b\int_0^\zeta (\frac{y}{b})^3\log(\frac{y}{b})dy = \frac{b^2}{4} \cdot (\frac{\zeta}{b})^4(\log(\frac{\zeta}{b}) - \frac{1}{4}) \quad . \tag{A.17}$$

Furthermore, by making the transformation $w = 1 + (1-(y/b)^2)^{1/2}$ , we obtain

$$b\int_0^\zeta (\frac{y}{b})^3\log(1 + (1-(\frac{y}{b})^2)^{1/2})dy =$$

$$b^2 \int_{1+(1-(\zeta/b)^2)^{1/2}}^{2} [(w-1)-(w-1)^3]\log w\, dw = \frac{1}{6} - \frac{1}{6}(1-(\frac{\zeta}{b})^2)^{1/2}(1 + \frac{1}{2}(\frac{\zeta}{b})^2) + \frac{1}{4}(\frac{\zeta}{b})^4(\log(1+(1-(\frac{\zeta}{b})^2)^{1/2}) - \frac{1}{4}) \tag{A.18}$$

Finally, substituting (A.15) - (A.18) in (A.14) we arrive at the formula

$$E\{Y^2;\zeta\} = b^2\{\frac{1}{6} - \frac{1}{6}(1-(\frac{\zeta}{b})^2)^{1/2}(1 + \frac{1}{2}(\frac{\zeta}{b})^2) - \frac{1}{4}\log((\zeta/b)/(1 + (1-(\frac{\zeta}{b})^2)^{1/2})\} \quad . \tag{A.19}$$

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The problem of determining the moments of visibility measures on star-shaped curves, under Poisson random shadowing process in the plane (see [4]) is generalized here to the problem of determining the moments of visibility measures on line segments in a three-cimensional space, when the shadowing process is created by a Poisson random field of spheres of random size. The present paper provides the methodology for reducing the three-dimensional problem to a two-cimensional one, and employing the general theory of [4] to solve the present problem. Although the general approach is similar, the

functions derived in the present paper are somewhat different for the sake of simplifying the computations.